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BUCKLING LOAD OF NON-LINEAR SYSTEMS WITH MULTIPLE EIGENVALUES

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Abstract—For structural systems with a coincident lowest eigenvalue λ_c, the influence of imperfections on the buckling of the systems depends to a very large extent upon the distribution of the imperfections. Moreover, the system may buckle either at a limit point or at a bifurcation point before this limit point is reached. Considering both possibilities, a lower bound to the buckling load of the system, for a given root mean square of the imperfections, is obtained. Furthermore, with reference to a set of particular, normalized co-ordinates, it was found that the absolute minimum buckling load is given by an imperfection vector parallel to the steepest of all post-buckling paths intersecting at λ_c. At this absolute minimum buckling load the critical point is a limit point. As an example, the lower bound to the buckling load of an imperfect cylindrical shell under axial compression was calculated.

1. INTRODUCTION

In structural mechanics, it is common practice to approximate the behaviour of a continuum by some discretization processes. The potential energy of a conservative structural system is then a function of the loading parameter λ and a finite set of generalized co-ordinates q_i. In most cases, the potential energy of the structure includes only linear terms in λ and in subsequent discussions, attention will be concentrated on such systems.

In linear analysis, the equilibrium path of a structure is governed by a set of linear simultaneous algebraic equations, the lowest eigenvalue of which gives the lowest buckling load of the structure. If this lowest eigenvalue λ_c of the linear system is distinct, the orientation of the corresponding eigenvector is uniquely defined. This eigenvector will in turn define uniquely the buckling mode of the structure. If nonlinear effects are to be included, the buckling load of the structure will obviously be different. For most structures, the most significant nonlinear terms may be represented by a so-called "imperfection vector". The change in buckling load, as is well known,[1] is directly related to the magnitude and orientation of this imperfection vector and also to the slope (or curvature) at λ_c of the uniquely defined post-buckling path. If the postbuckling path is steep, reduction in buckling load will be large. Maximum reduction is obtained when the imperfection vector is parallel to the eigenvector corresponding to the lowest eigenvalue λ_c. The critical point of the nonlinear system referred to in the above discussion is a limit point, that is, a local maximum point on the nonlinear basic equilibrium path of the structure.

For the particular case where λ_c is not distinct but is, for instance, an m-fold eigenvalue, the eigenvectors will not be uniquely defined and there exist more than one post-buckling path in the neighbourhood of λ_c[1, 4, 5]. If reduction in buckling load were again influenced by the slope (or curvature) of the post-buckling path, one would then expect that the equilibrium path of the nonlinear system closest to the post-buckling path with the steepest slope (or maximum curvature) will have the least buckling load. The imperfection vector which produces such a basic equilibrium path will hence be the "worst" imperfection vector. With reference to a particular set of normalized coordinates u_i and defining the "worst" imperfection vector as one which produces the "absolute minimum buckling load" for a given magnitude of the normalized imperfection parameter ε, this has in fact been proved to be true for two typical categories of structural systems. Moreover, in the proof given, the possibility of bifurcation buckling was also considered. With the "absolute minimum buckling load" thus established, it was then possible to obtain a lower bound to the buckling load of the system for a given root mean square of the imperfections.

Finally, to demonstrate quantitatively the above ideas, the typical example of a thin imperfect circular cylinder under axial compression was studied. It is well known that a perfect, long cylinder may buckle into many different wave forms at the same buckling load λ_c. In other words the eigenvalue of the linear system is a multiple eigenvalue. The worst imperfection vector and the corresponding lower bound to the buckling load was found. As was expected, the buckling load given by Koiter[2] who assumes arbitrarily that the imperfection is axisymmetrical is greater than the estimated lower bound.

2. LIMIT POINTS AND BIFURCATION POINTS

Consider a structural system of m degrees of freedom with a potential energy given as follows:†

$$V(q_i, \lambda, \kappa) = \lambda_{(i)}B_{ij}q_iq_j + \lambda C_{ij}q_iq_j + B_{ijk}q_iq_jq_k + \lambda\kappa D_{(i)}q_id_i \tag{1a}$$

where $i = 1$ to m, q_i are the generalized co-ordinates and λ and κ are the loading and imperfection parameters respectively. The vector d_i represents the imperfection of the structure and the root mean square Q of the imperfection can be given in terms of d_i as follows:

$$Q = M\{a^2_{(i)}d_id_i\}^{1/2}$$

where $a_{(i)}$ are known constants. Note that the summation convention applies to all Latin subscripts not in brackets.

For a given value of the root mean square of the imperfections, the vector d_i is normalized by the following condition:

$$a^2_{(i)}d_id_i = 1.$$

For the particular case where the eigenvalues $\lambda_{(i)}$ are coincident such that $\lambda_{(i)} = \lambda_c$ for all i, the quadratic forms B_{ij} and C_{ij} will be diagonalized for any set of co-ordinates q_i. Equation (1a) then takes the following form:

$$V(q_i, \lambda, \kappa) = (\lambda_c - \lambda)B^2_{(i)}q_iq_i + B_{ijk}q_iq_jq_k + \lambda\kappa D_{(i)}q_id_i. \tag{1b}$$

In equation (1b), we shall assume that λ takes positive values only.

Rewriting the above in terms of a set of normalized co-ordinates u_i, we have

$$V(u_i, \lambda, \varepsilon) = A^2\{(\lambda_c - \lambda)u_iu_i + A_{ijk}u_iu_ju_k + \lambda\varepsilon S_iu_i\} \tag{1c}$$

† For the more general case of a system with n degrees of freedom having a m-fold lowest eigenvalue ($m < n$), as a first approximation it is necessary to consider only the effect of the first m "critical" co-ordinates, see Koiter[1].

where

$$u_i = B_{(i)} q_i / A$$
$$A_{ijk} = AB_{ijk}/(B_{(i)} B_{(j)} B_{(k)})$$
$$\beta S_i = \frac{D_{(i)}}{AB_{(i)}} d_i \quad \text{and} \quad S_i S_i = 1$$
$$\varepsilon = \kappa\beta.$$

The constant A is chosen such that $\beta \leqslant 1$.

Considering equation (1c) we see that an orthogonal transformation $u_i = \alpha_{ij}\bar{u}_j$ with $\alpha_{i1} = S_i$ exists such that $\bar{S}_i = S_j \alpha_{ji} = \delta_{i1}$ where δ_{ij} is the Kronecker delta[6], i.e.

$$V(u_i, \lambda, \varepsilon) = A^2\{(\lambda_c - \lambda)\bar{u}_i\bar{u}_i + \bar{A}_{ijk}\bar{u}_i\bar{u}_j\bar{u}_k + \lambda\varepsilon\bar{u}_1\} \tag{2}$$

where

$$\bar{A}_{ijk} = A_{rst}\alpha_{ri}\alpha_{sj}\alpha_{tk}$$

in particular

$$\bar{A}_{111} = A_{rst} S_r S_s S_t.$$

The behaviour of the structure depends obviously upon the relative magnitude of the coefficients $\bar{A}_{ijk}$ and in particular upon the magnitude of $\bar{A}_{111}$. The orientation of the imperfection vector S_i therefore bears a direct influence upon the behaviour of the structure. Our purpose is to seek the optimum orientation of S_i which will produce the maximum $\bar{A}_{111}$.

To find a lower bound to the buckling load, it is necessary to consider the possibility of both snap through at a limit point and bifurcation buckling. Conditions which govern these two different types of buckling behaviour are well known[7]. However, for the clarity of subsequent discussion it is perhaps worthwhile to present a brief derivation of these conditions.

Consider for example a structure with a potential energy given by equation (1c). The equilibrium of the structure then requires,

$$2(\lambda_c - \lambda)u_i + 3A_{ijk}u_j u_k + \lambda\varepsilon S_i = 0. \tag{3}$$

For critical stability,

$$\det|V_{ij}| = 0, \tag{4}$$

where partial differentiation with respect to u_i, u_j etc. is represented by, $_{ij}$, etc.

Let us now perturb the governing equilibrium equations with respect to a suitably chosen independent parameter z, starting from a point (λ^0, u_i^0) on the equilibrium path. Differentiating equation (3) once with respect to z, we have,

$$B_{rs}u_s' = \lambda' C_r \tag{5}$$

where

$$B_{rs} = 2(\lambda_c - \lambda^0)\delta_{rs} + 6A_{rst}u_t^0$$
$$C_r = 2u_r^0 - \varepsilon S_r \qquad (r, s, t = 1 \text{ to } m).$$

Differentiation with respect to z is represented by an apostrophe.

At $\lambda^0 = \bar{\lambda}$, the buckling load of the structure, $\det|B_{rs}| = 0$. Hence for all $C_r \neq 0$, a non-trivial solution u'_s exists if and only if $\lambda' = 0$, that is, the system buckles only at a local maximum point on the basic equilibrium path. Consider however the case when some but not all of the coefficients C_r vanish. For example when $S_p = 0$ and $A_{pij} = 0$ for all $p > h$ and $i, j \leqslant h$ it follows that $u_i^0 \neq 0$, $u_p^0 = 0$ is a solution of equations (3). Equation (5) can then be separated into two sets of decoupled equations,

$$D_{ij} u'_j = \lambda' C_i$$
$$E_{pq} u'_q = \lambda' C_p \qquad (i, j \leqslant h; p, q > h)$$

and

$$C_p = 2u_p^0 - \varepsilon S_p = 0.$$

At a critical point, $\det|B_{rs}| = \det|D_{ij}| . \det|E_{pq}| = 0$; several possibilities therefore exist and are described below:

(i) $\det|D_{ij}| = 0$, $\det|E_{pq}| \neq 0$, for a non-trivial solution u'_j to exist λ' must vanish. The critical point is therefore a limit point.

(ii) $\det|D_{ij}| \neq 0$, $\det|E_{pq}| = 0$, hence at least two equilibrium paths exist, that is, $u'_q = 0$ and $u'_q \neq 0$. The intersection of these two paths is then a bifurcation point.

(iii) $\det|D_{ij}| = 0$, $\det|E_{pq}| = 0$. For a non-trivial solution u'_j to exist λ' must vanish. However, since $\det|E_{pq}| = 0$, it is again possible for two equilibrium paths $u'_q = 0$ and $u_i^0 \neq 0$ to exist. The intersection of these two paths is now a limit point.

Besides the cases listed above, bifurcation may also occur if the determinant $|D_{ij}|$ and the set of coefficients C_i vanish simultaneously at a particular loading λ^0, that is, when the solution vector u_i^0 becomes parallel to the imperfection vector S_i. However, for the particular case where $S_i = 0$ except for $i = 1$, λ^0 is negative and the solution is therefore of no immediate interest to us.

Summing up, for $S_r = \delta_{r1}$, the necessary conditions for the existence of bifurcation points are:

$$(1)\ A_{pij} = 0 \qquad (i, j \leqslant h, p > h)$$
$$(2)\ \det|E_{pq}| = 0. \tag{6}$$

3. SOME INEQUALITIES

Consider the optimization of the function $\bar{A}_{\alpha\beta..\xi} = A_{ijk..n} t_{i\alpha} t_{j\beta} t_{k\gamma} .. t_{n\xi}$ where $t_{i\alpha}$, $t_{j\beta}$, etc. are arbitrarily orientated m-dimensional unit vectors. Note that summation convention does not apply to the Greek suffices. Let $t_{i\alpha}^+$, $t_{j\beta}^+$, etc. be vectors which optimize $\bar{A}_{\alpha\beta..\xi}$, then they must satisfy the following equations:

$$A_{ijk..n} t_{j\beta} t_{k\gamma} .. t_{n\xi} - 2\phi_1 t_{i\alpha} = 0 \tag{7a}$$

$$A_{ijk..n} t_{j\alpha} t_{k\gamma} .. t_{n\xi} - 2\phi_2 t_{i\beta} = 0, \tag{7b}$$

etc.

and

$$t_{i\alpha} t_{i\alpha} = t_{j\beta} t_{j\beta} = \cdots = t_{n\xi} t_{n\xi} = 1$$

where ϕ_1, $\phi_2 \ldots$, etc. are the Lagrange multipliers and $2\phi_1 = 2\phi_2 = \cdots = A_{ijk..n} t_{i\alpha} t_{j\beta} t_{k\gamma} \ldots t_{n\xi} = A_{\alpha\beta..\xi}$. Adding equations (7a) and (7b), we have,

$$(A_{ijk..n} t_{k\gamma}..t_{n\xi} - 2\phi_2 \delta_{ij}) t_{j\beta} + (A_{ijk..n} t_{k\gamma}..t_{n\xi} - 2\phi_1 \delta_{ij}) t_{j\alpha} = 0.$$

Again from equation (7a),

$$(A_{ijk..n} t_{k\gamma}..t_{n\xi} - 2\phi_1 \delta_{ij}) t_{j\beta} - 2\phi_1 (t_{i\alpha} - t_{i\beta}) = 0.$$

Hence if $\det|A_{ijk..n} t_{k\gamma}..t_{n\xi} - 2\phi_1 \delta_{ij}| \neq 0$, then $t_{j\beta} = -t_{j\alpha}$.

If on the other hand the determinant vanishes, then a non-trivial solution exists if and only if $\phi_1 = 0$ or $t_{i\alpha} = t_{i\beta}$. Similarly, it can be shown that $|t^+_{i\beta}| = |t^+_{i\gamma}| = \cdots = |t^+_{i\xi}|$. In other words, at the non-zero optimums of $A_{\alpha\beta..\xi}$ the vectors $|t^+_{i\alpha}|$, $|t^+_{i\beta}|$, etc. must be identical. Furthermore, let $A^+_{\alpha\beta.\xi}$ be the local optimums of $A_{\alpha\beta..\xi}$ and $|A^*_{111.1}|$ the global or absolute maximum of the function $|A_{ijk..n} S_i S_j S_k S_n|$,.. where S_i represents an arbitrarily orientated unit initial imperfection vector, then $|\bar{A}_{\alpha\beta..\xi}| \leqslant |A^+_{\alpha\beta..\xi}| \leqslant |A^*_{111.1}|$ for all $1 \leqslant \alpha, \beta, ..\xi \leqslant m$.

To find $A^+_{111.1}$, note that S^+_i which optimizes the function $A_{ijk..n} S_i S_j S_k..S_n$ with the constraint $S_i S_i = 1$ must satisfy the following $m + 1$ equations:

$$N A_{ijk..n} S_j S_k..S_n - 2\phi S_i = 0 \tag{8a}$$

$$S_i S_i = 1 \tag{8b}$$

where ϕ is the Lagrange multiplier. Multiplying equation (8a) with the transformation vector α_{ir} and sum over the index i, we obtain,

$$N A_{ijk..n} S_j S_k..S_n \alpha_{ir} = 2\phi S_i \alpha_{ir}.$$

Since $\alpha_{i1} = S_i$ and the transformation is orthogonal, that is, $\alpha_{ir}\alpha_{is} = \delta_{rs}$, we have,

$$\bar{A}_{111..1s} = A_{ijk..n} S^+_j S^+_k..S^+_n \alpha_{is} = 0 \qquad s \neq 1$$

that is $\bar{A}_{111..1s}$ vanishes when $\bar{A}_{111..1}$ attains its local optimum.

Let us now compare equations (3) with (8a) where N is now equal to 3. For a linear system, $\varepsilon = 0$ in (3) and if u_i is a solution to (3) with $\varepsilon = 0$, then it is obvious that $S_i = \gamma u_i$ is a solution of (8a). To find γ, note that $S_i S_i = 1$ hence $\gamma^2(u_i u_i) = 1$ or $\gamma = (u_i u_i)^{-1/2}$. Hence the optimum vectors S^+_i are parallel to the projections onto the normalized u_i subspace of the post buckling paths of the structure. Moreover, since the slope of the postbuckling path is given by the magnitude of A_{111}, the imperfection vector S^*_i which yields the global maximum A^*_{111} will be parallel to the post-buckling path with the greastest slope.

4. SNAP BUCKLING

To find a lower bound to the snap buckling load, consider the first of the equilibrium equations (3),

$$2(\lambda_c - \lambda)u_1 + 3A_{1jk} u_j u_k + \lambda\varepsilon S_1 = 0$$

where $S_1 \neq 0$. Expanding the above, we have, for $s, r \neq 1$

$$(3A_{111})u_1^2 + [2(\lambda_c - \lambda) + 6A_{11s} u_s]u_1 + (3A_{1sr} u_s u_r + \lambda\varepsilon S_1) = 0$$

hence

$$u_1 = \frac{-(\lambda_c - \lambda) + 3A_{11s} u_s \pm \{[(\lambda_c - \lambda) + 3A_{11s} u_s]^2 - (3A_{111})(3A_{1sr} u_s u_r + \lambda\varepsilon S_1)\}^{1/2}}{3A_{111}}.$$

At $\lambda = \bar{\lambda}$ the snap buckling, the equilibrium path reaches a local maximum point, that is, the terms within the bracket in the above equation cancel each other. The two solutions of the quadratic equation coincide and are given by,

$$\bar{u}_1 = -\frac{(\lambda_c - \bar{\lambda}) + 3A_{11s}\bar{u}_s}{3A_{111}} \tag{9}$$

where $\bar{u}_i$ gives the equilibrium position of the structure at $\bar{\lambda}$. Equation (9) can be rewritten as follows:

$$(\lambda_c - \bar{\lambda}) = 3A_{11i}\bar{u}_i \qquad i = 1 \text{ to } m$$

or

$$|\lambda_c - \bar{\lambda}| = 3|\bar{u}|\,|A_{11i}a_i|$$

where $\bar{u} = (\bar{u}_i\bar{u}_i)^{1/2}$ is the magnitude of the vector $\bar{u}_i$ and $a_i = \bar{u}_i/\bar{u}$ is a unit vector. Moreover, the sum $A_{11i}a_i$ can be rewritten as follows:

$$A_{11i}a_i = A_{jkl}t_{j1}t_{k1}t_{i\theta}$$

where $t_{j1} = \delta_{j1}$ and $t_{i\theta} = a_i$. Hence it is obvious that $|A_{11i}a_i|$ must be less than or equal to the global maximum $|A^*_{111}|$.
that is

$$|\lambda_c - \bar{\lambda}| \leqslant 3|\bar{u}|\,|A^*_{111}|. \tag{10}$$

To find the magnitude u of the vector u_i, multiply the ith equilibrium equation by u_i and sum over the index i,

$$2(\lambda_c - \lambda)u^2 + 3u^3(A_{ijk}a_i a_j a_k) + \lambda\varepsilon u S_i a_i = 0.$$

For $u \neq 0$, we have,

$$Au^2 + Bu + C = 0 \tag{11a}$$

where

$$A = 3A_{ijk}a_i a_j a_k$$

is bounded

$$B = 2(\lambda_c - \lambda) > 0$$
$$C = \lambda\varepsilon S_i a_i.$$

Let b_1, b_2 where $|b_1| \leqslant |b_2|$ be the two real roots of (11a), such that

$$(u - b_1)(u - b_2) = 0$$

then

$$b_1 b_2 = \frac{C}{A} \tag{11b}$$

$$b_1 + b_2 = -\frac{B}{A}. \tag{11c}$$

At a limit point, $b_1 = b_2$, that is $B^2 - 4AC = 0$. Hence for a limit point to exist A and C must be of the same sign. From equations (11b) and (11c) it follows therefore that both b_1 and b_2 are of opposite sign to A.

$$|b_1||b_2| = \left|\frac{C}{A}\right|$$

and

$$|b_1| + |b_2| = \left|\frac{B}{A}\right|.$$

From the above two equations we have,

$$|b_1|^2 - |b_1||b_2| = \left|\frac{B}{A}\right||b_1| - \left|\frac{2C}{A}\right| \leqslant 0$$

or

$$|b_1| \leqslant \left|\frac{2C}{B}\right|$$

that is

$$|u| \leqslant \left|\frac{\lambda\varepsilon S_i a_i}{\lambda_c - \lambda}\right| \leqslant \left|\frac{\lambda\varepsilon}{\lambda_c - \lambda}\right|. \tag{12}$$

The above inequality is valid for all $0 \leqslant \lambda < \lambda_c$.
At $\lambda = \bar{\lambda}$, equations (12) and (10) then give

$$(\lambda_c - \bar{\lambda})^2 \leqslant |3A^*_{111}\bar{\lambda}\varepsilon|. \tag{13}$$

Substituting for $\varepsilon = \kappa\beta \leqslant \kappa$ (since $\beta \leqslant 1$), we have,

$$(\lambda_c - \bar{\lambda})^2 \leqslant |3A^*_{111}\bar{\lambda}\kappa|.$$

Defining λ^* by $(\lambda_c - \lambda^*)^2 = |3A^*_{111}\lambda^*\kappa|$, it is obvious that $\bar{\lambda} \leqslant \lambda^*$. λ^* is therefore a lower bound to the snap buckling load of the structure.

In terms of the normalised co-ordinates, equation (13) can be interpreted geometrically as follows: if an imperfection vector S^*_i can be found such that after the orthogonal transformation $u_i = \alpha^*_{ij}\bar{u}^*_j$ where $\alpha^*_{i1} = S^*_i$, the coefficient $\bar{A}^*_{111} = A_{ijk}\alpha^*_{i1}\alpha^*_{j1}\alpha^*_{k1}$ attains its global maximum A^*_{111} and $\bar{S}^*_i = \delta_{i1}$, then $\bar{A}^*_{11s}$ vanishes for all $s \neq 1$. The sth equilibrium equation therefore gives $\bar{u}^*_s = 0$. From equation (4), the corresponding snap buckling load is then given by $(\lambda_c - \bar{\lambda}^*)^2 = 3A^*_{111}\bar{\lambda}^*\varepsilon$. Comparing this result with equation (13), it is obvious that $\bar{\lambda} \leqslant \bar{\lambda}^*$. In other words for a given value of the imperfection parameter ε the snap buckling load $\bar{\lambda}^*$ corresponding to an impervection vector S^*_i (where S^*_i is parallel to the post-buckling path with the greatest slope A^*_{111}) is the absolute minimum snap buckling load of the structure.

5. BIFURCATION BUCKLING

From Section 2, we see that for $S_p = 0$ a bifurcating point exists if and only if the coefficients A_{pij} vanish for all $p > h$ and $i, j \leqslant h$. Equations (3) then separate into two sets of decoupled equations as follows:

$$2(\lambda_c - \lambda)u_i + 3A_{ijk}u_j u_k + \lambda\varepsilon S_i = 0 \tag{14a}$$

$$u_p = 0 \qquad p > h \quad \text{and} \quad i, j, k \leqslant h. \tag{14b}$$

The critical point is now defined by,

$$\det|2(\lambda_c - \lambda)\delta_{pq} + 6A_{pqi}u_i| = 0 \qquad p, q > h. \tag{15}$$

Note that the coefficients $B_{pq} = A_{pqi}u_i$ are invariant with respect to any transformation in u_i. Let $\bar{u}_i$ be the solution of equation (14a) at $\lambda = \bar{\bar{\lambda}}$, then a transformation in the u_p coordinates can always be found such that $\bar{B}_{pq} = \bar{A}_{pqi}\bar{u}_i = \delta_{pq}$. It is necessary to consider only bifurcation points with buckling loads $\bar{\bar{\lambda}}$ less than λ_c, hence at least one of the coefficients $\bar{B}_{pq}$ must be negative. Let $\bar{B}_{\theta\theta} = A_{\theta\theta i}\bar{u}_i$ be the least of all negative $\bar{B}_{pq}$, then the bifurcation buckling load $\bar{\bar{\lambda}}$ is given as follows:

$$(\lambda_c - \bar{\bar{\lambda}}) + 3\bar{A}_{\theta\theta i}\bar{u}_i = 0$$

or

$$(\lambda_c - \bar{\bar{\lambda}}) = |\bar{u}|\,|3\bar{A}_{\theta\theta i}a_i| \tag{16}$$

where again $a_i = \bar{u}_i/\bar{u}$ is a unit vector.

Consider firstly the bifurcation point which is located on an equilibrium path with a local maximum (point II, Fig. 1). Equation (12) is still valid, that is, at $\lambda = \bar{\bar{\lambda}}$

$$|\bar{u}| \leqslant \left|\frac{\bar{\bar{\lambda}}_c}{\lambda_c - \bar{\bar{\lambda}}}\right|.$$

Hence for $|\bar{A}_{\theta\theta i}a_i| \leqslant |A^*_{111}|$, we obtain,

$$(\lambda_c - \bar{\bar{\lambda}})^2 \leqslant 3|A^*_{111}\bar{\bar{\lambda}}\varepsilon| \leqslant |3A^*_{111}\bar{\bar{\lambda}}\kappa|.$$

From the above inequality, it is obvious that $\bar{\bar{\lambda}} \leqslant \bar{\lambda}^*$ and $\bar{\bar{\lambda}} \leqslant \lambda^*$.

Besides the situation discussed above, bifurcation points may also exist on equilibrium paths without maximums (e.g. point III, Fig. 1). The coefficients A and C in equation (11a)

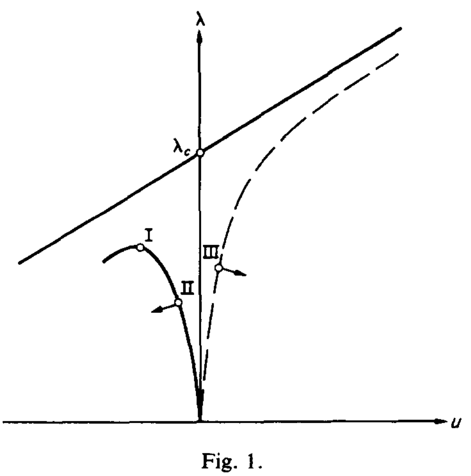

Fig. 1.

are now of opposite sign. Let b_1 and b_2 where $|b_2| > |b_1|$ be the two roots of (11a) then from equation (11c), it is obvious that for $A < 0, b_1 < 0$ and for $A > 0$, $b_1 > 0$. Substituting for b_1, the quadratic equation can be written as follows,

$$|b_1|^2 + \left|\frac{B}{A}\right| |b_1| - \left|\frac{C}{A}\right| = 0$$

that is

$$\left|\frac{B}{A}\right| |b_1| < \left|\frac{C}{A}\right| \qquad \text{for } b_1 \neq 0$$

$$|b_1| < \left|\frac{C}{B}\right|$$

hence

$$|\bar{\bar{u}}| < \left|\frac{\bar{\bar{\lambda}}\varepsilon}{2(\lambda_c - \bar{\bar{\lambda}})}\right|.$$

Substituting for $\bar{\bar{u}}$, we have therefore

$$(\lambda_c - \bar{\bar{\lambda}})^2 < \tfrac{3}{2}|A^*_{111}\bar{\bar{\lambda}}\varepsilon|$$

that is, $\bar{\bar{\lambda}}$ is again greater than $\bar{\lambda}^*$ and λ^*. Hence, with reference to the normalized co-ordinates, $\bar{\lambda}^*$ is the absolute minimum of all possible buckling loads produced by imperfection vectors with the same magnitude ε but different orientations. Moreover, $\bar{\lambda}^*$ is the buckling load corresponding to an imperfection vector parallel to the postbuckling path with the steepest slope. Note however, that the normalized imperfection parameter ε has no direct physical meaning. For example, the root mean square of different imperfections with the same ε will in general be different, (see example in Section 7). Meanwhile, comparing the effect of different imperfections with the same root mean square, λ^* given above is a lower bound to all the possible buckling loads.

6. FOURTH ORDER SYSTEMS

In the previous sections, discussion was concentrated upon systems with potential energy V given by equation (1a). However, similar conclusions can easily be drawn for systems with nonlinear terms of the fourth order.

If the more important terms only were considered, the potential energy of a fourth order system can be written in the normalized co-ordinates as follows:

$$V(u_i, \lambda, \varepsilon) = \{(\lambda_c - \lambda)u_i u_i + A_{ijkl} u_i u_k u_j u_l + \lambda\varepsilon S_i u_i\}A^2. \tag{17}$$

The equilibrium paths of the structure are then defined by the following equations,

$$2(\lambda_c - \lambda)u_i + 4A_{ijkl} u_j u_k u_l + \lambda\varepsilon S_i = 0. \tag{18}$$

Multiplying the ith equation with u_i and sum over the index i, we obtain,

$$2(\lambda_c - \lambda)u^2 + (4A_{ijkl} a_i a_j a_k a_l)u^4 + \lambda\varepsilon u S_i a_i = 0$$

where $u^2 = u_i u_i$. Since $u \neq 0$, the above equation then becomes,

$$Au^3 + Bu + C = 0 \tag{19}$$

where

$$A = 4A_{ijkl}a_i a_j a_k a_l$$
$$B = 2(\lambda_c - \lambda) > 0$$
$$C = \lambda \varepsilon S_i a_i.$$

Let b_1, b_2, b_3 be the three roots of equation (19) such that,

$$(u - b_1)(u - b_2)(u - b_3) = 0$$

then

$$b_1 b_2 b_3 = -C/A \tag{20a}$$

$$b_1 b_2 + b_1 b_3 + b_2 b_3 = B/A \tag{20b}$$

$$b_1 + b_2 + b_3 = 0. \tag{20c}$$

If a local maximum point exists on the basic equilibrium path, two of the three roots must be real and equal. Hence from equation (20c), $2b_1 = -b_3$. Substituting this result into equations (20a) and (20b), we have, at the local maximum point $\lambda = \bar{\lambda}$,

$$b_1^3 = C/2A \quad \text{and} \quad b_1^2 = -B/3A$$

the existence of a local maximum then requires,

$$(C/2A)^2 = (-B/3A)^3.$$

Substituting for A, B and C, we have therefore,

$$(\lambda_c - \bar{\lambda})^3 = -(3/2)^3(A_{ijkl}a_i a_j a_k a_l)(\bar{\lambda}\varepsilon S_i a_i)^2.$$

Since only the solution $\bar{\lambda} < \lambda_c$ is of interest, the right hand side can be assumed to take positive values only. Moreover, from Section 3 $|A_{ijkl}a_i a_j a_k a_l| \leqslant |A^*_{1111}|$, hence,

$$(\lambda_c - \bar{\lambda})^3 \leqslant (3/2)^3 |A^*_{1111}|(\bar{\lambda}\varepsilon)^2 \leqslant (\tfrac{3}{2})^3 |A^*_{1111}|(\bar{\lambda}\kappa)^2. \tag{21}$$

Let λ^* be defined as follows:

$$(\lambda_c - \lambda^*)^3 = (3/2)^3 |A^*_{1111}|(\lambda^*\kappa)^2. \tag{22}$$

It is obvious that $\bar{\lambda}$ will always be greater than λ^*.

For fourth order systems, the necessary conditions for the existence of bifurcation points are the same as those given in equation (6), except that A_{pijk} instead of A_{pij} must now vanish. From condition (6), we have, therefore, for bifurcation buckling,

$$\det|2(\lambda_c - \bar{\bar{\lambda}})\delta_{pq} + B_{pq}| = 0$$

where $B_{pq} = 12A_{pqij}u_i u_j$ with $p, q > h$ and $i, j \leqslant h$. Again a tranformation $u_p = \beta_{pq}\bar{u}_q$ among the u_p coordinates can always be found such that the above determinant is diagonalized. The lowest bifurcation load is then given by,

$$(\lambda_c - \bar{\bar{\lambda}}) = -\tfrac{1}{2}\bar{B}_{\theta\theta} = -6\bar{A}_{\theta\theta ij}\bar{u}_i\bar{u}_j = -(6\bar{A}_{\theta\theta ij}a_i a_j)(\bar{u})^2 \tag{23}$$

where $\bar{A}_{\theta\theta ij} = A_{pqij}\beta_{p\theta}\beta_{q\theta}$. $\bar{u}_i$ is a solution for $\lambda = \bar{\bar{\lambda}}$ of the first set of the now decoupled equilibrium equations (18) and $a_i = \bar{u}_i/\bar{u}$ is a unit vector. The magnitude $\bar{u}$ of $\bar{u}_i$ is again governed by a cubic equation of the same form as equation (19), except that indices i, j, k, l

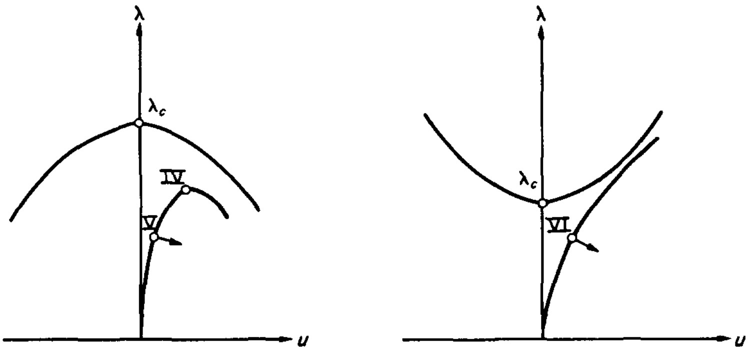

Fig. 2.

are now summed from 1 to h instead of from 1 to m. For the present case the cubic equation may have either (a) three real roots, or (b) one real root only.

Consider firstly case (a). Let $|b_1| < |b_2| < |b_3|$ be the three real roots. Then from equation (20c), b_1 and b_2 must be of the same sign. Substituting equation (20c) into (20a) and (20b) we have,

$$b_1^2 b_2 + b_1 b_2^2 = C/A \tag{24a}$$

$$b_1^2 + b_1 b_2 + b_2^2 = -B/A. \tag{24b}$$

From the last equation, it is obvious that a real solution exists for $\lambda < \lambda_c$ if and only if $A < 0$. It therefore follows from equation (24a) that b_1, b_2 and C are of opposite sign. Equation (24a) together with (24b) then gives,

$$2b_1^3 - b_1 b_2(b_1 + b_2) = -2Bb_1/A - 3C/A.$$

Hence for $C > 0$, $b_1 < 0$,

$$-2\left|\frac{B}{A}\right| |b_1| + 3\left|\frac{C}{A}\right| > 0$$

for $C < 0$, $b_1 > 0$,

$$2\left|\frac{B}{A}\right| |b_1| - 3\left|\frac{C}{A}\right| < 0$$

that is

$$|b_1| < |3C/2B|$$

or

$$|u| < |3\lambda\varepsilon S_i a_i/4(\lambda_c - \lambda)|$$
$$|u| < |3\lambda\varepsilon/4(\lambda_c - \lambda)|.$$

The above inequality is true for all $0 < \lambda < \lambda_c$. From equation (23) it is obvious that $(\lambda_c - \bar{\bar{\lambda}})^3 < (3/2)^3 |A^*_{1111}| (\bar{\bar{\lambda}}\varepsilon)^2$ that is $\lambda^* < \bar{\bar{\lambda}}$.

Consider next case (b). Let b_1 be the only real root then, the constant A is now positive. Equation (19) then gives,

$$b_1^3 + \frac{B}{A} b_1 + \frac{C}{A} = 0.$$

From the above equation, it is clear that for all $\lambda < \lambda_c$, b_1 and C must be of opposite sign. Hence, for $C > 0$, $b_1 < 0$,

$$-|b_1|^3 - \left|\frac{B}{A}\right| |b_1| + \left|\frac{C}{A}\right| = 0$$

for $C < 0$, $b_1 > 0$

$$|b_1|^3 + \left|\frac{B}{A}\right| |b_1| - \left|\frac{C}{A}\right| = 0$$

hence

$$\left|\frac{C}{A}\right| = \left|\frac{B}{A}\right| |b_1| + |b_1|^3 > \left|\frac{B}{A}\right| |b_1|$$

that is

$$|b_1| < |C/B|.$$

The bifurcation load is then bounded as follows:

$$(\lambda_c - \bar{\bar{\lambda}})^3 < (3/2)|A^*_{1111}|(\bar{\bar{\lambda}}\varepsilon)^2.$$

From the above result, it is again obvious that $\bar{\bar{\lambda}}$ must be greater than λ^*. In other words, λ^* is a lower bound to all possible buckling loads for a given root mean square of the imperfections. Furthermore, an imperfection vector S^*_i parallel to the post-buckling path with the maximum curvature will produce a snap buckling $\bar{\lambda}^*$, where $(\lambda_c - \bar{\lambda}^*)^3 = (3/2)^3|A^*_{1111}| \cdot (\bar{\lambda}^*\varepsilon)^2$. From the above discussion, it is obvious that for a given magnitude of imperfection parameter ε, $\bar{\lambda}^*$ is the absolute minimum buckling load of the system.

7. EXAMPLE

The foregoing analysis can now be applied to study the behaviour of an imperfect long cylindrical shell under axial compression. For this purpose, the formulation of the problem given by Koiter in Refs. [1, 2] is accepted. Koiter assumes that the radial displacement w of the shell is given by,

$$w = fR + c_0 \sin p_0 \frac{x}{R} + \sum_k \left(a_{k1} \sin p_{k1} \frac{x}{R} + c_{k2} \cos p_{k2} \frac{x}{R}\right) \cos k\theta \qquad (25)\dagger$$

where x and θ are co-ordinates measured on the undeformed mid-surface of the shell along the axial and circumferential directions respectively. The pre-buckling uniform radial expansion of the cylinder is given by the fR where R is the radius of the shell. The buckling mode meanwhile is described by the coefficients c_0, a_{k1} and c_{k2} and p_{k1}, p_{k2} which are the two roots of the following equation:

$$p^2 - p_0 p + k^2 = 0$$

where

$$p_0 = [12(1 - \nu^2)^{1/4}]\left(\frac{R}{h}\right)^{1/2}.$$

† Summation convention does not apply to equations (25)–(28).

In the above equation, k is the number of circumferential waves and p the corresponding axial wave number. The thickness of the cylinder is given by h and v is Poisson's ratio. Substituting the expression for w into the equilibrium equations of the shell, the axial and tangential displacements u and v of the cylinder can be determined.

The initial deformations of the cylinder can again be expressed in terms of $\cos \kappa\theta$, $\sin p_{k1}\frac{x}{R}$ and $\cos p_{k2}\frac{x}{R}$ as follows:

$$w_0 = (\mu h)\left[C_0 \sin p_0 \frac{x}{R} + \sum_k \left(A_{k1} \sin p_{k1} \frac{x}{R} + C_{k2} \cos p_{k2} \frac{x}{R}\right) \cos k\theta\right] \tag{26†}$$

where the coefficients C_0, and A_{k1} C_{k2} are dimensionless and μh gives the magnitude of w_0.

Substituting for w, u, v and w_0 in the potential energy function of the cylinder[1] we have,

$$V(c_0, a_{k1}, c_{k2}, \lambda, \mu h) = \frac{Eh}{2v^2}(f^2 - 2f\lambda v)(2\pi RL)$$
$$+ \frac{\pi EhL}{4R}\{(\lambda_c - \lambda)[2p_0^2 c_0^2 + \sum_k (p_{k1}^2 a_{k1}^2 + p_{k2}^2 c_{k2}^2)] + 3\frac{c_0}{R}\sum_k k^2 a_{k1} c_{k2}$$
$$-2\lambda(\mu h)[2p_0^2 c_0 C_0 + \sum_k (p_{k1}^2 a_{k1} A_{k1} + p_{k2}^2 c_{k2} C_{k2})]\} \tag{27†}$$

where $\lambda = \frac{\sigma}{E}$ is the loading parameter, σ the applied compression stress, E Young's modulus, L length of the cylinder and $\lambda_c = \left(\frac{h}{R}\right)/\sqrt{3(1 - v^2)}$ the buckling load of the perfect cylinder.

Writing, $\varepsilon = \left(2\mu \frac{h}{R}\right) \cdot \beta = \kappa\beta \qquad k^2 = p_{k1}p_{k2}$

$$u_1 = \frac{c_0}{R} \qquad u_k = \frac{1}{\sqrt{2}}\frac{a_{k1}p_{k1}}{p_0 R} \qquad u_{(k+n)} = \frac{1}{\sqrt{2}}\frac{c_{k2}p_{k2}}{p_0 R}$$

$$\beta S_1 = C_0 \qquad \beta S_k = \frac{1}{\sqrt{2}}\frac{A_{k1}p_{k1}}{p_0} \qquad \beta S_{(k+n)} = \frac{1}{\sqrt{2}}\frac{C_{k2}p_{k2}}{p_0}$$

$$\beta^2 = \left[C_0^2 + \sum \frac{1}{2}\left(\frac{p_{k1}}{p_0}\right)^2 A_{k1}{}^2 + \sum \frac{1}{2}\left(\frac{p_{k2}}{p_0}\right)^2 C_{k2}{}^2\right]$$

$1 < k \leqslant n + 1$‡ (28)†

we have,

$$V(\lambda, \varepsilon, f, u_i) = \frac{Eh}{2v^2}(f^2 - 2f\lambda v)(2\pi RL)$$
$$+ \frac{\pi hREL}{2}p_0^2[(\lambda_c - \lambda)u_i u_i + 3u_1 u_k u_{(k+n)} - \lambda\varepsilon S_i u_i] \tag{29}$$

where $1 < k \leqslant n + 1$, $1 \leqslant i \leqslant 2n + 1$.

‡ To satisfy the assumption that boundary conditions at both ends of the cylinder can be ignored, k must not be too small.

The equilibrium paths of the cylinder are defined by the equation $f = \lambda v$ together with the following equations:

$$2(\lambda_c - \lambda)u_1 + 3u_k u_{(k+n)} - \lambda \varepsilon S_1 = 0 \tag{30a}$$

$$2(\lambda_c - \lambda)u_k + 3u_1 u_{(k+n)} - \lambda \varepsilon S_k = 0 \tag{30b}$$

$$2(\lambda_c - \lambda)u_{(k+n)} + 3u_1 u_k - \lambda \varepsilon S_{(k+n)} = 0 \tag{30c}$$

where $1 < k \leqslant n+1$, $\varepsilon = \kappa\beta$.

For $\varepsilon = 0$, the solutions of the above equations are,

(i) $u_1 = u_k = u_{(k+n)} = 0$ which represents the trivial basic equilibrium path.

(ii) $u_k = \pm\, u_{(k+n)}$ for all k and $u_1 = \mp \frac{2}{3}(\lambda_c - \lambda)$, in particular, u_k, $u_{(k+n)}$ may vanish for some or all except one value of k.

From equation (30a),

$$2(\lambda_c - \lambda)u_1 = \pm\, 3u_k u_k = \pm\, 3u_{(k+n)}u_{(k+n)}$$

hence

$$u_1 u_1 = u_k u_k = u_{(k+n)}u_{(k+n)}.$$

The optimum imperfection vectors S_i^+ is given by,

$$S_i^+ = \gamma u_i$$

where

$$S_i^+ S_i^+ = \gamma^2 u_i u_i = \gamma^2(u_1 u_1 + u_k u_k + u_{(k+n)(k+n)})$$

or

$$\gamma = \frac{1}{\sqrt{3u_1}}$$

and

$$\bar{A}_{111} = A_{rst} S_r^+ S_s^+ S_t^+ = \gamma^3(A_{rst} u_r u_s u_t)$$

but

$$\begin{aligned} A_{rst} &= 3 \quad \text{for} \quad r = 1,\ 2 \leqslant s \leqslant n+1, \quad \text{and} \quad t = s+n. \\ &= 0 \quad \text{for all other values of } r,\ s,\ t \end{aligned}$$

hence

$$\bar{A}_{111} = \gamma^3(3u_s u_{(s+n)}u_1) = \gamma^3(3u_1^3) = 1/\sqrt{3}.$$

Since all possible combinations of the antisymmetrical solutions u_k, $u_{(k+n)} \neq 0$ give the same value of $1/\sqrt{3}$ for $\bar{A}_{111}$, the global maximum A_{111} is therefore equal to $1/\sqrt{3}$. λ^* is now given by,

$$(\lambda_c - \lambda^*)^2 = (3A_{111}^* \lambda^* \kappa) = 2\sqrt{3}\lambda^* \mu(h/R). \tag{31}$$

The root mean square of the initial deflection w_0 of the cylinder is given as follows:

$$\begin{aligned} Q^2 &= \int_0^L \int_0^{2\pi} (R w_0^2\, \mathrm{d}x\, \mathrm{d}\theta) \cdot (2\pi LR)^{-1} \\ &= \tfrac{1}{2}(\mu h)^2(C_0^2 + \tfrac{1}{2}A_{k1}A_{k1} + \tfrac{1}{2}C_{k2}C_{k2}). \end{aligned}$$

Note that $C_0 = d_1$, $A_{k1} = d_k$ and $C_{k2} = d_{(k+n)}$ for $1 < k \leqslant n+1$. Hence, for $Q^2 = \frac{1}{2}(\mu h)^2$, the vector d_i is normalized by the following condition:

$$C_0^2 + \tfrac{1}{2}A_{k1}A_{k1} + \tfrac{1}{2}C_{k2}C_{k2} = 1.$$

Since p_{k1}/p_0 and p_{k2}/p_0 are both less than one, comparing the above condition with equation (28), it is obvious that $\beta < 1$. Hence, for an initial deflection of root mean square $\mu h/\sqrt{2}$, equation (31) then gives a lower bound to all possible buckling loads.

For example if the initial deformation is axisymmetrical, i.e. $C_0 \neq 0$ $A_{k1} = C_{k2} = 0$ for all k, then for a root mean square of $\mu h/\sqrt{2}$, $C_0 = 1$, $\beta = 1$. The buckling load given by Koiter[1] is as follows:

$$(\lambda_c - \bar{\bar{\lambda}})^2 = 1.5\bar{\bar{\lambda}}\mu(h/R)$$

$\bar{\bar{\lambda}}$ is obviously greater than the lower bound λ^*.

The relation between Q and ε for this particular case is given by

$$Q = \frac{1}{\sqrt{2}}(\mu h)C_0 = \frac{1}{\sqrt{2}}(\mu h)\beta = \frac{R\varepsilon}{2\sqrt{2}}.$$

Again, assume an imperfection vector $S_1 = S_k = S_{(k+n)} = 1/\sqrt{3}$, with $p_{k1} = p_{k2} = \frac{1}{2}p_0$. For $Q = \frac{1}{\sqrt{2}}(\mu h)$ we have $C_0 = 1/3$, $A_{k1} = C_{k2} = 2\sqrt{2}/3$ and $\beta = \sqrt{3}C_0 = \frac{1}{\sqrt{3}}$. The buckling load of the cylinder is then given by

$$(\lambda_c - \bar{\lambda})^2 = 3\bar{A}_{111}\bar{\lambda}\kappa\beta = 2\bar{\lambda}\mu(h/R)$$

which is again greater than the lower bound given in equation (31). Finally for given ε, the root mean square of the imperfections is as follows:

$$Q = \frac{1}{\sqrt{2}}(\mu h)(3C_0) = \sqrt{\frac{3}{2}}(\mu h)\beta = \frac{1}{2}\sqrt{\frac{3}{2}}R\varepsilon.$$

Hence, it is obvious that the root mean square depends not only on ε but also on the orientation of the imperfection vector d_i.

8. CONCLUSION

A lower bound to the buckling load of an imperfect structural system with an m-fold eigenvalue was established. Referring to a set of particular, normalized co-ordinates u_i, it was found that if the imperfection vector is orientated along the direction of the post-buckling path with the steepest slope or maximum curvature, the buckling load of the system will attain its global minimum. This result can be considered as a generalization of Roorda's [3] conclusion on major and minor imperfections for systems with distinct eigenvalues. Rigorous proof was given for two typical categories of structural systems. In the analysis, the only restriction imposed on the imperfection vector S_i is the constraint $S_iS_i = 1$. In other words, the imperfection vector was assumed to be physically feasible for all possible variation of $0 \leqslant S_i \leqslant 1$. ($i = 1$ to m). Apart from this assumption, the proof given was completely general.

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